Atmosphere: #4
Equation of motion

1. Geostrophic motion

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = \mathscr{F}_x$$

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = \mathscr{F}_y$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

Are all of these terms important every time?

1. Geostrophic motion

U/T U^2/L   *f*U

$$\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = 0$$

$$\frac{\partial v}{\partial t} + \mathbf{u}\cdot\nabla v + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0$$

1/*f*T   U/(*f*L)   1

$$R_o = \frac{U}{fL}$$ → Rossby number

- First, we consider flows where friction can be neglected.
- Then, let's guess how big each term can be.

1. Geostrophic motion

U/T U2/L fU

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = 0$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0$$

1/fT U/(fL) 1

$$R_o = \frac{U}{fL} \rightarrow \text{Rossby number}$$

- First, we consider flows where friction can be neglected.
- Then, let's guess how big each term can be.
- For typical large-scale flows in the atmosphere:

- U ~ 10 m s^{-1} (horizontal velocity scale)
- W ~ 1 cm s-1 (vertical velocity scale)
- L ~ 10^6 m (length scale)
- T ~ 10^5 s (time scale)
- f ~ 10^{-4} s^{-1}
- $\frac{1}{\rho}\frac{\partial p}{\partial x} \sim 10^{-3}$

1. Geostrophic motion

$$R_o = \frac{U}{fL} \longrightarrow$$ Rossby number ~ 10^{-1}

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - fv = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0$$

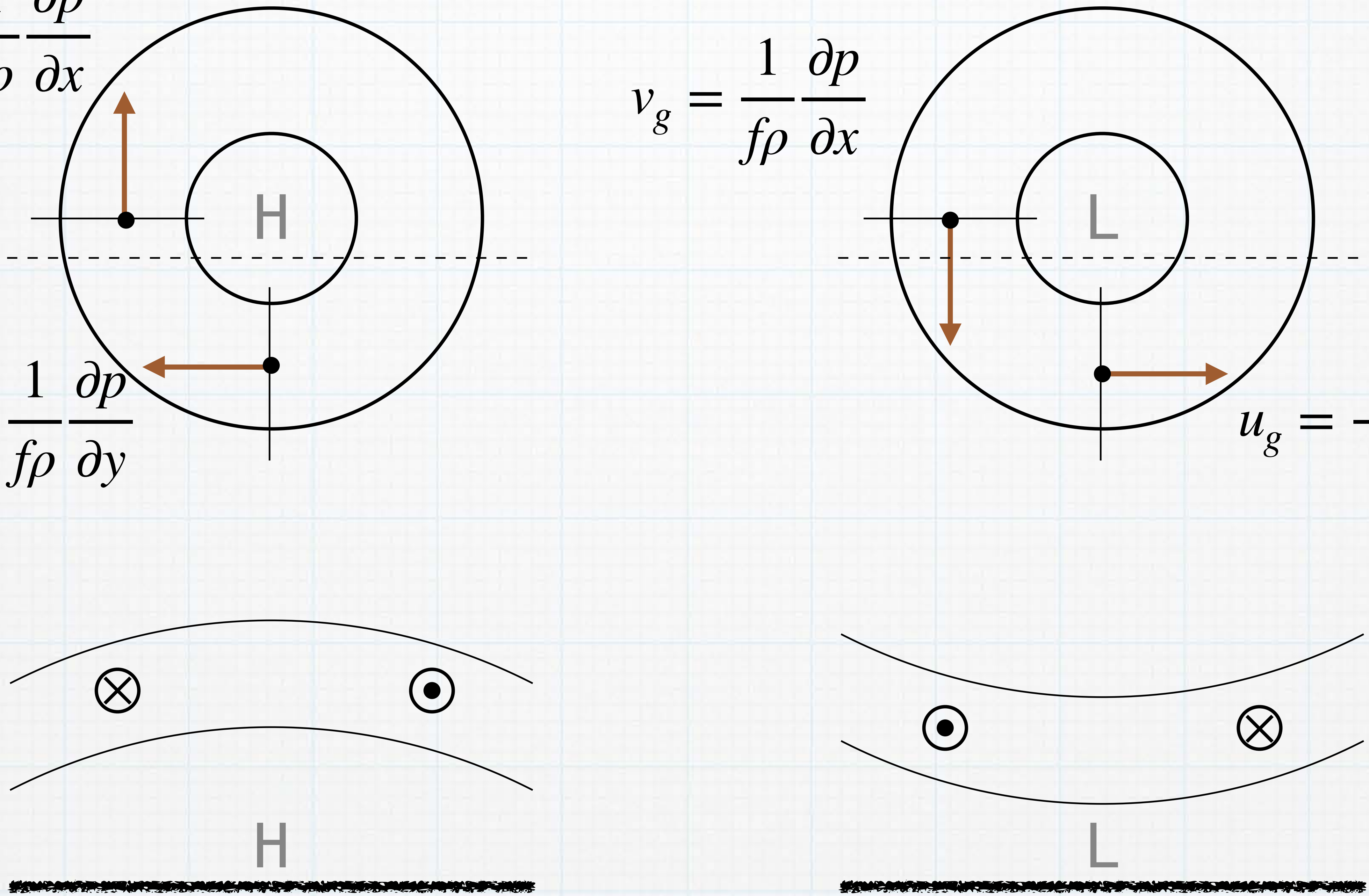
Geostrophic balance

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

Geostrophic wind

1. Geostrophic motion



$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

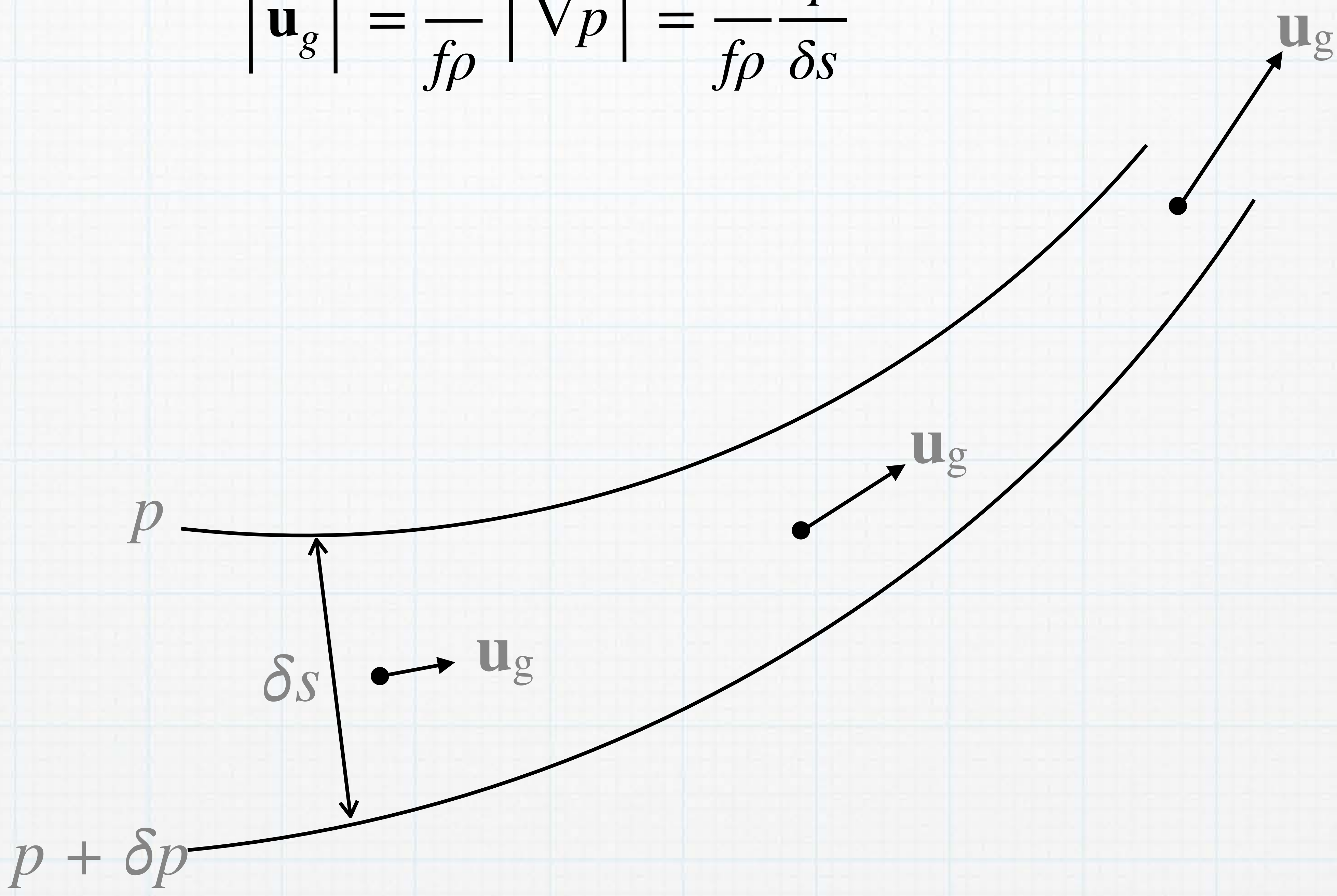
$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

1. Geostrophic motion

$$\left|\mathbf{u}_g\right| = \frac{1}{f\rho}\left|\nabla p\right| = \frac{1}{f\rho}\frac{\delta p}{\delta s}$$



1. Geostrophic motion : Nondivergent flow

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = -\frac{1}{f\rho}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{f\rho}\frac{\partial^2 p}{\partial x \partial y} = 0$$

Non divergent flow:

any change in u_g will be compensated by the change in v_g

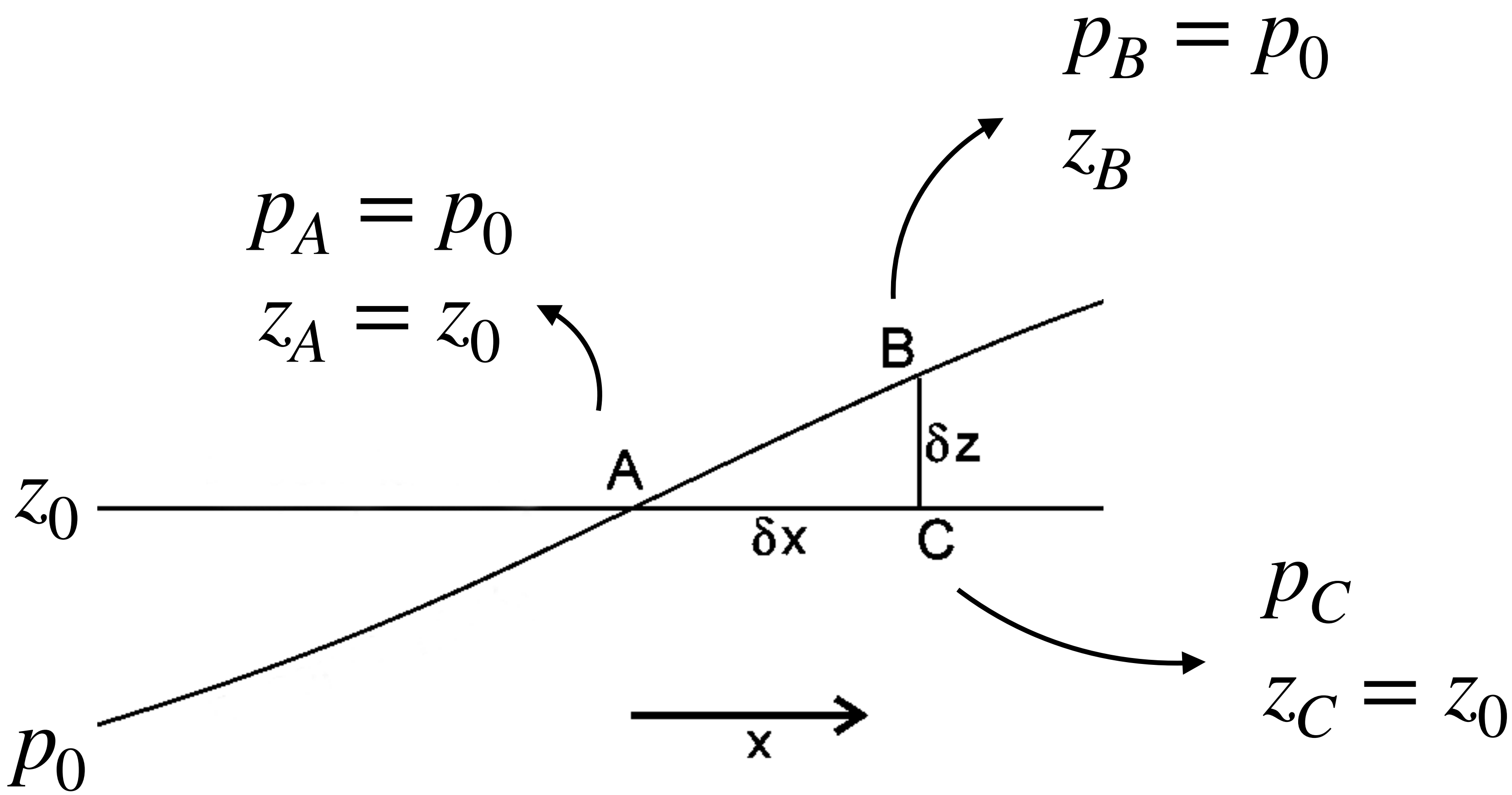
$$\rightarrow \frac{\partial w_g}{\partial z} = 0$$

→ if w_g=0 on a flat bottom boundary, then w_g=0 everywhere!

→ in this case, the geostrophic flows is horizontal.

1. Geostrophic motion : pressure coordinates

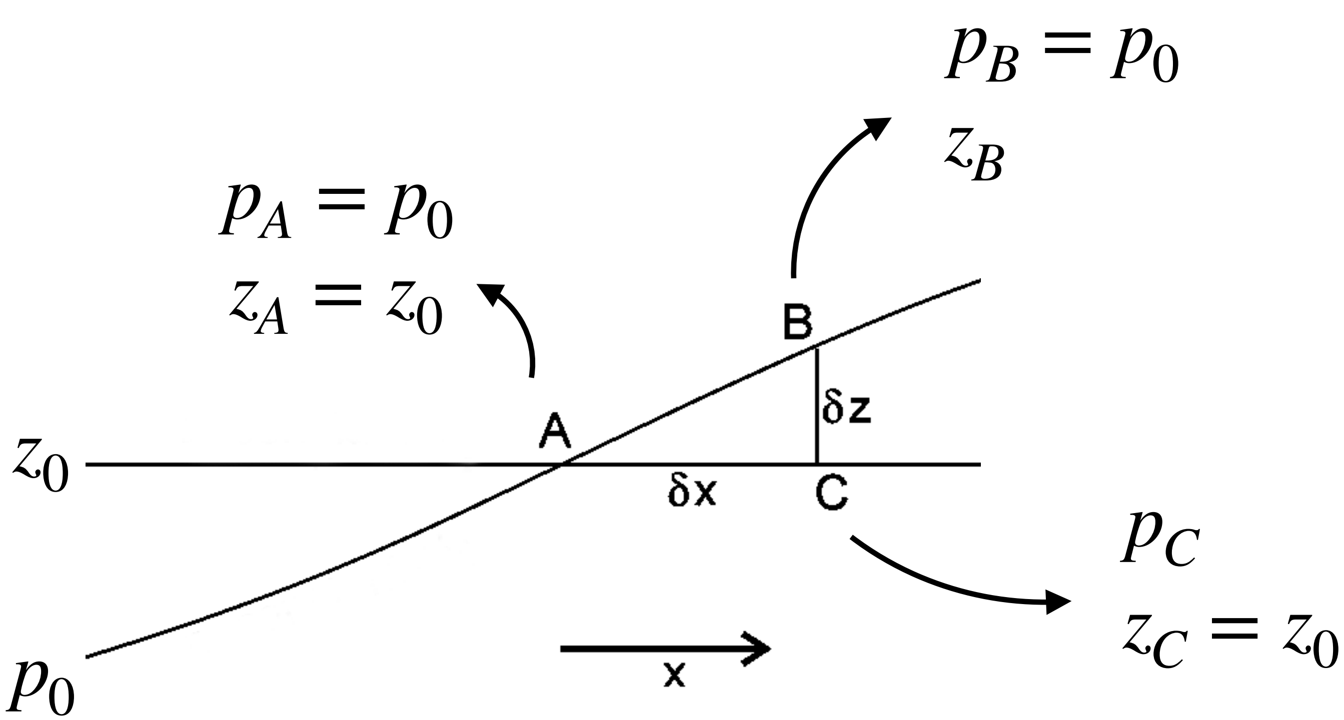
- It is convenient to look at the geostrophic wind on the pressure coordinate.



Pressure gradient between C and A : $\left(\frac{\partial p}{\partial x}\right)_z = \frac{p_C - p_0}{\delta x}$

The gradient of height between B and A : $\left(\frac{\partial z}{\partial x}\right)_p = \frac{z_B - z_0}{\delta x}$

1. Geostrophic motion : pressure coordinates



$$\frac{p_C - p_0}{z_B - z_0} = \frac{p_C - p_B}{z_B - z_C} = -\frac{\partial p}{\partial z} = g\rho$$

$$\downarrow$$

$$p_C - p_0 = g\rho(z_B - z_0)$$

$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho\left(\frac{\partial z}{\partial x}\right)_p \qquad \left(\frac{\partial p}{\partial y}\right)_z = g\rho\left(\frac{\partial z}{\partial y}\right)_p$$

1. Geostrophic motion : pressure coordinates

$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho\left(\frac{\partial z}{\partial x}\right)_p$$

$$\left(\frac{\partial p}{\partial y}\right)_z = g\rho\left(\frac{\partial z}{\partial y}\right)_p$$

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

Lateral gradient of

Geopotential height

No ρ!

1. Geostrophic motion : pressure coordinates

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

If f is kept constant,

$$\nabla_p \cdot \mathbf{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} + \frac{\partial \omega_g}{\partial p} = -\frac{g}{f}\frac{\partial^2 z}{\partial x \partial y} + \frac{g}{f}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial \omega_g}{\partial p} = 0$$

$$\frac{\partial \omega_g}{\partial p} = 0$$

1. Geostrophic motion : pressure coordinates

- Introducing $\Psi_g = \Psi_g(x, y, p, t)$ which satisfies

$$u_g = -\frac{\partial \Psi_g}{\partial y} \text{ and } v_g = \frac{\partial \Psi_g}{\partial x}$$

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

- Then, $\Psi_g = \frac{g}{f}z$ and is known as a streamfunction.

- Geopotential height shows the streamline of the geostrophic flow.

1. Geostrophic motion : pressure coordinates

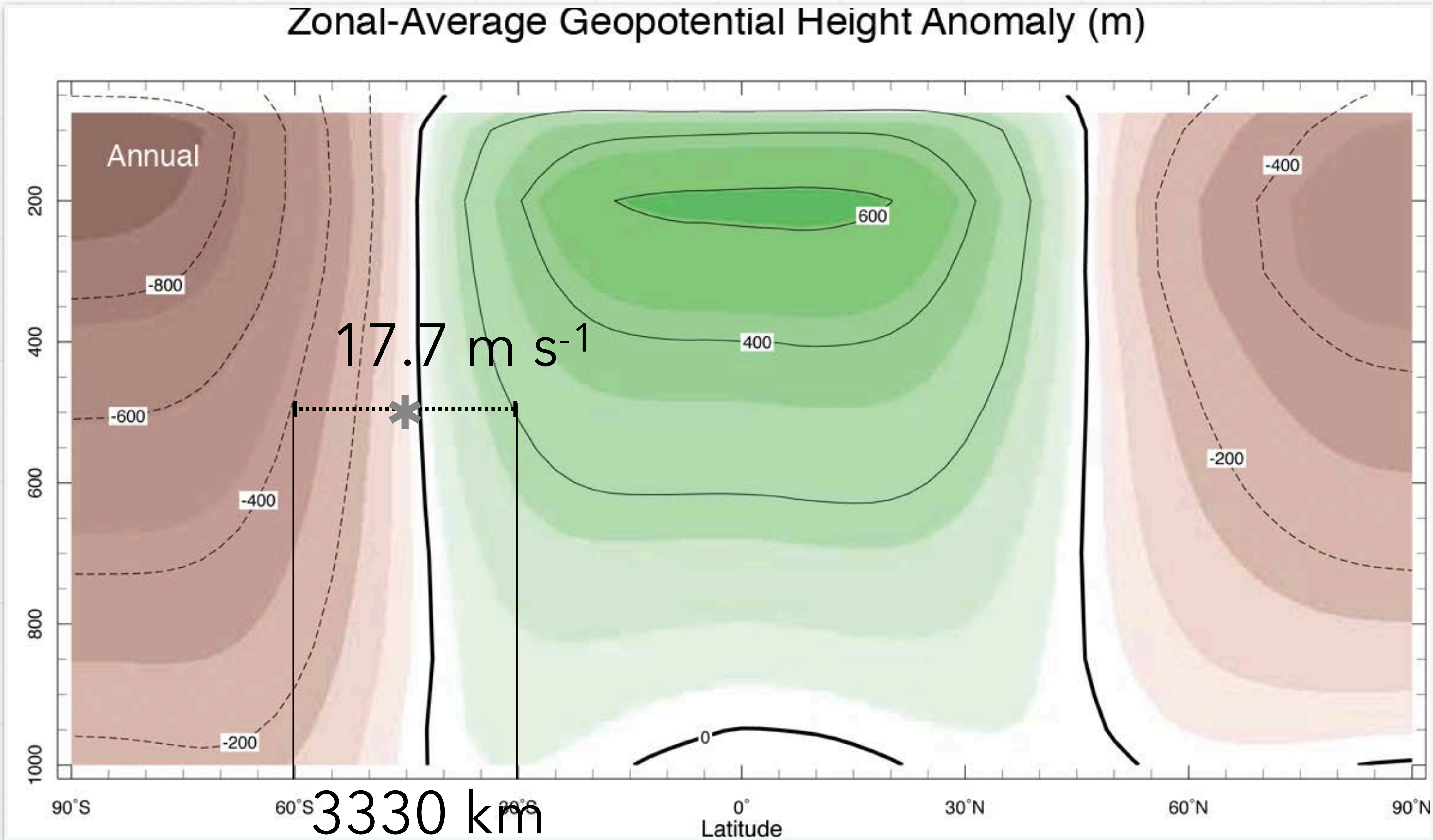


Figure 5.13, Marshall and Plumb (2008)

1. Geostrophic motion : pressure coordinates

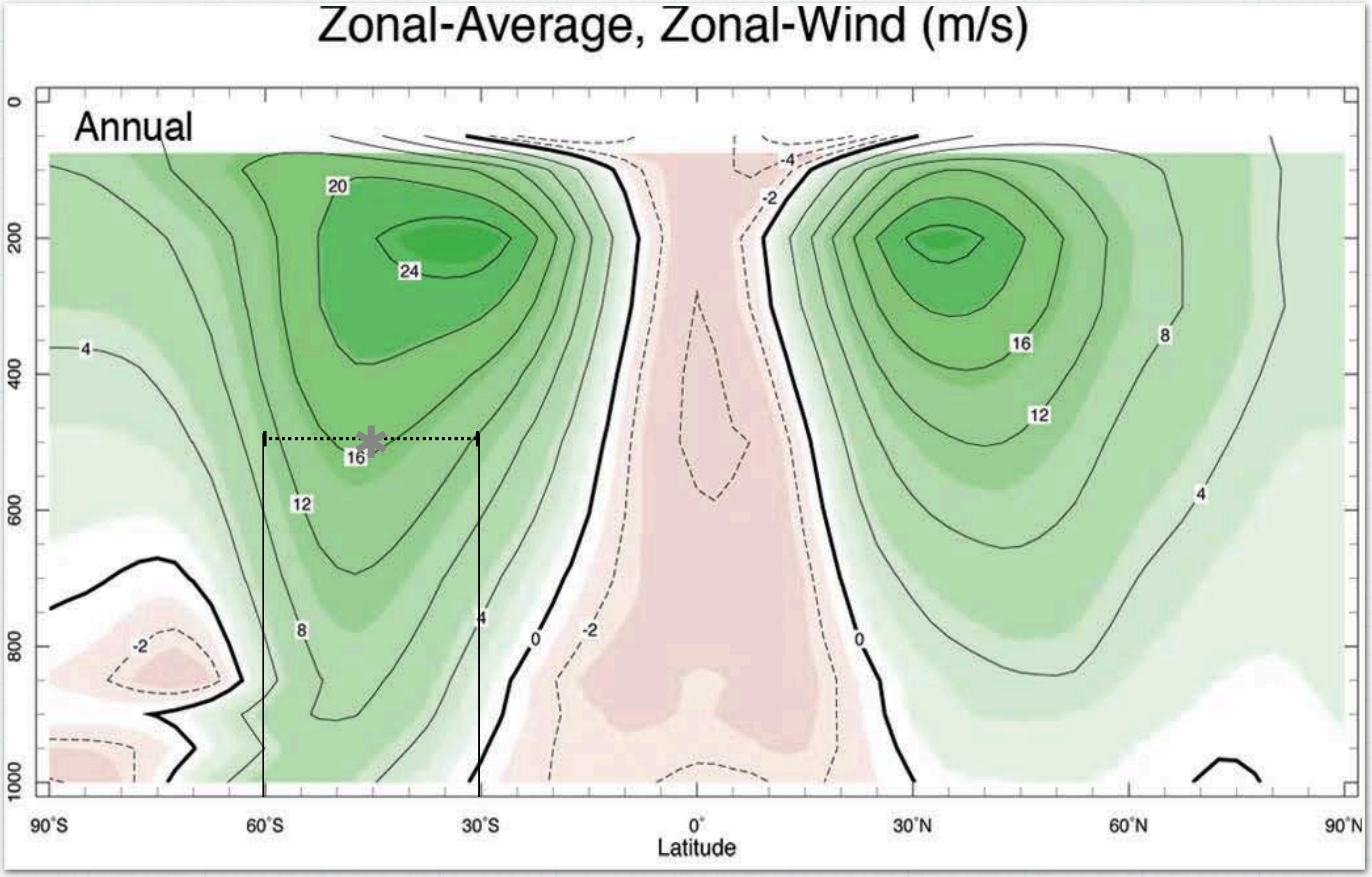
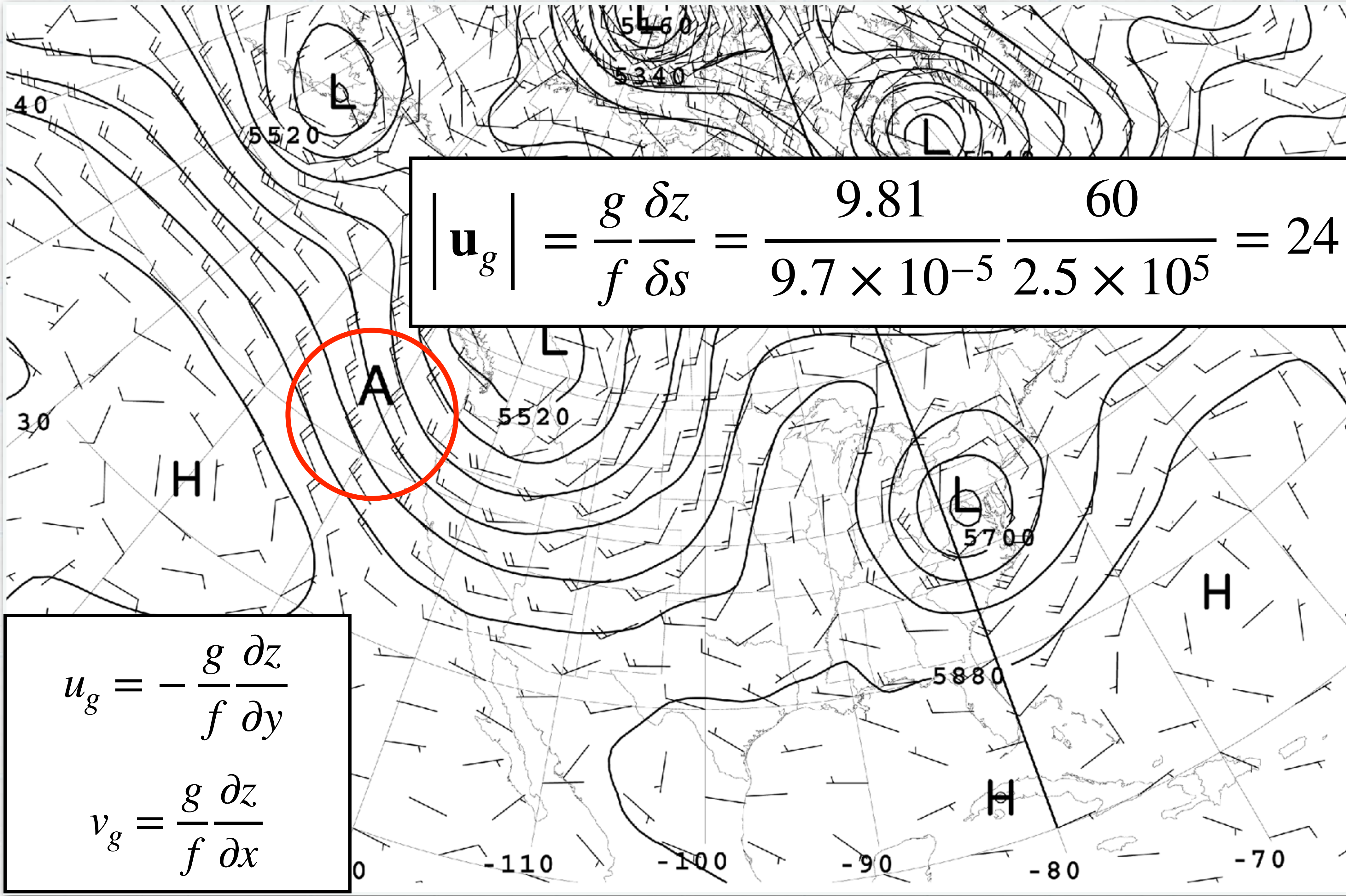


Figure 5.20, Marshall and Plumb (2008)

1. Geostrophic motion : pressure coordinates

The wind speed near A is approximately 25 m s^{-1}



$$\left|\mathbf{u}_g\right| = \frac{g}{f}\frac{\delta z}{\delta s} = \frac{9.81}{9.7\times10^{-5}}\frac{60}{2.5\times10^{5}} = 24 \text{ m s}^{-1}$$

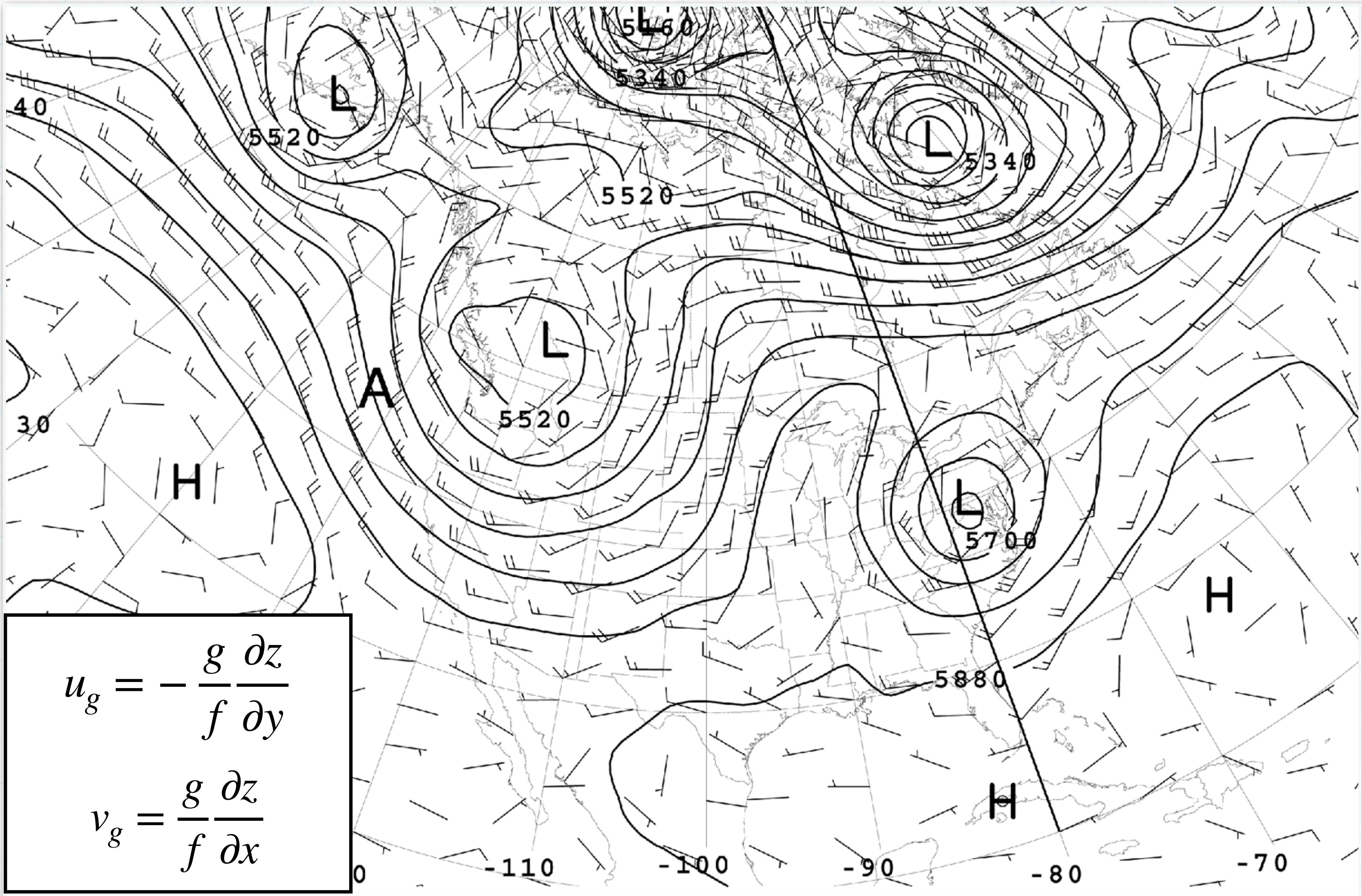
$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

Figure 7.4, Marshall and Plumb (2008)

1. Geostrophic motion : pressure coordinates

The strength of the geostrophic wind relies on the lateral gradient of geostrophic height.



$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

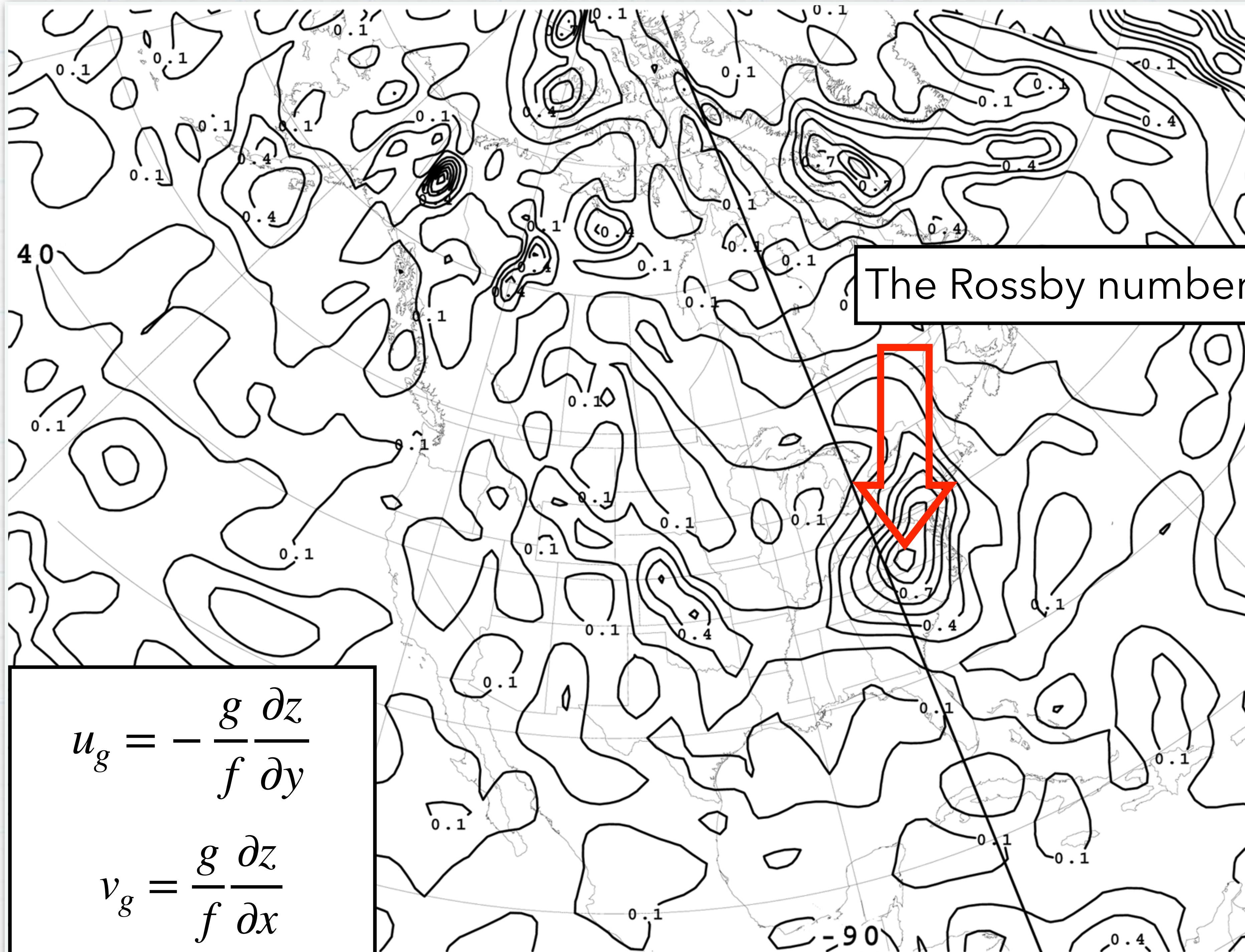
$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

Figure 7.4, Marshall and Plumb (2008)

1. Geostrophic motion : pressure coordinates

Rossby number

Figure 7.5, Marshall and Plumb (2008)



The Rossby number approaches 1

$$R_o = \frac{U}{fL}$$

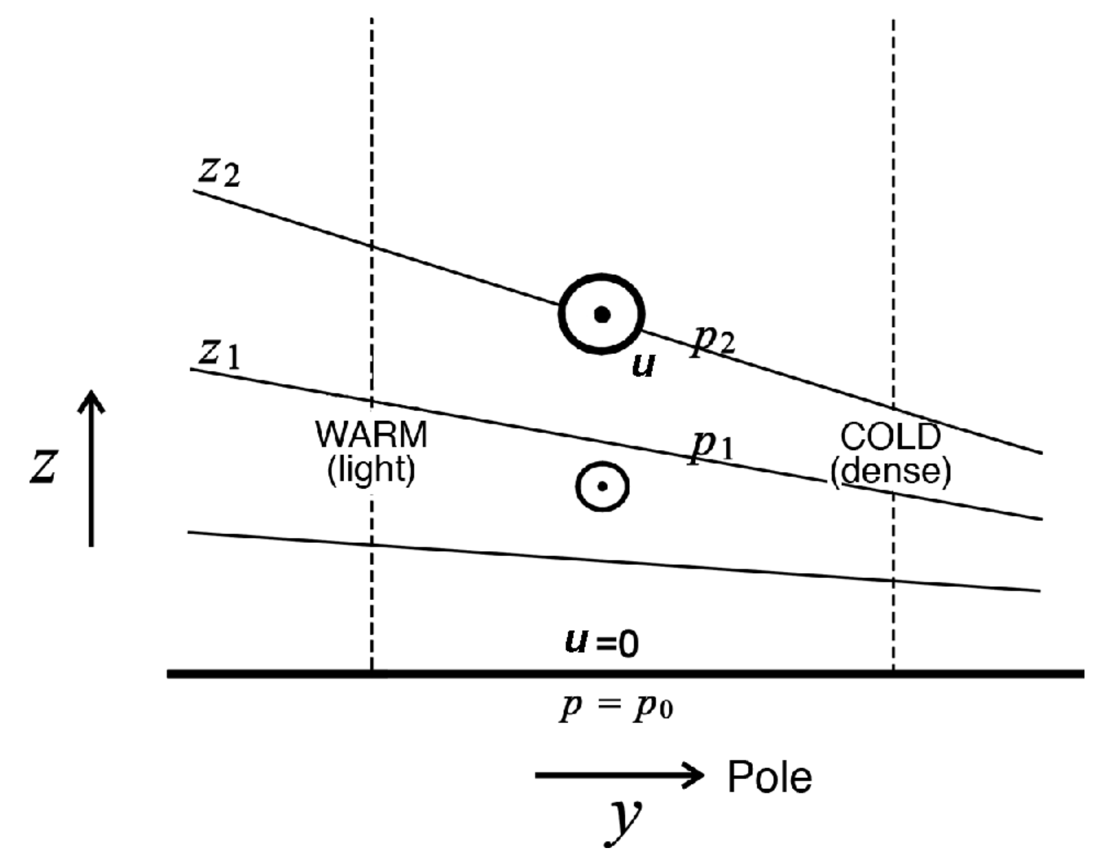
$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

Gradient wind balance

3. Thermal wind equation

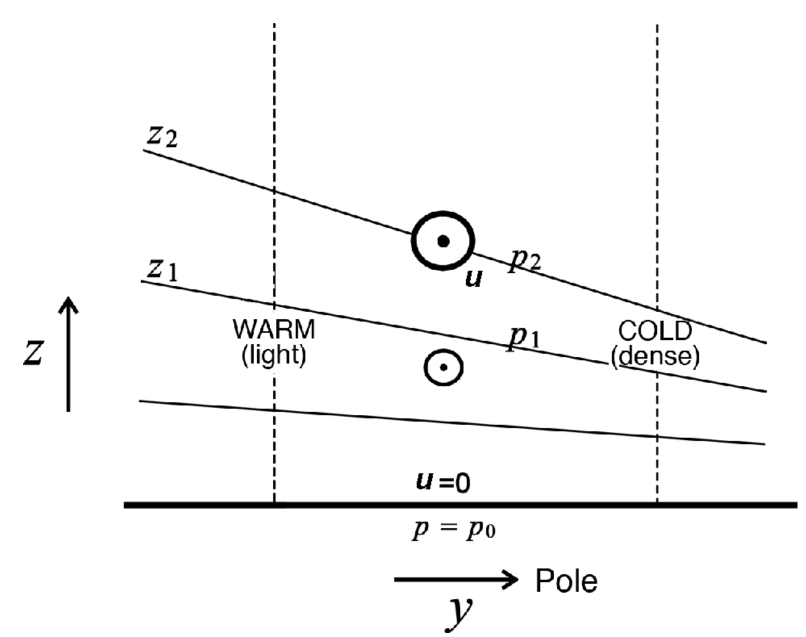
- The slopes of isobaric surfaces increase with height.
- According to the geostrophic relation, the geostrophic flow will increase with height.
- Contrast to the Taylor-Proudman theorem, the density varies in space.



3. Thermal wind equation

- Consider $\left(\dfrac{\partial u_g}{\partial z}, \dfrac{\partial u_g}{\partial z}\right)$ with $\rho = \rho_0 + \sigma$.
- $$\frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) = -\frac{1}{f\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = \frac{g}{f\rho_0}\frac{\partial \sigma}{\partial y}$$
- $$\frac{\partial v_g}{\partial z} = \frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) = \frac{1}{f\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = -\frac{g}{f\rho_0}\frac{\partial \sigma}{\partial x}$$

  Or
- $$\frac{\partial \mathbf{u}_g}{\partial z} = -\frac{g}{f\rho_0}\hat{\mathbf{z}} \times \nabla \sigma$$



3. Thermal wind equation

- In case of water, $\sigma \approx -\alpha T'$

Thermal expansion coefficient

$$\frac{\partial \mathbf{u}_g}{\partial z} = \frac{\alpha g}{f}\hat{\mathbf{z}} \times \nabla T'$$

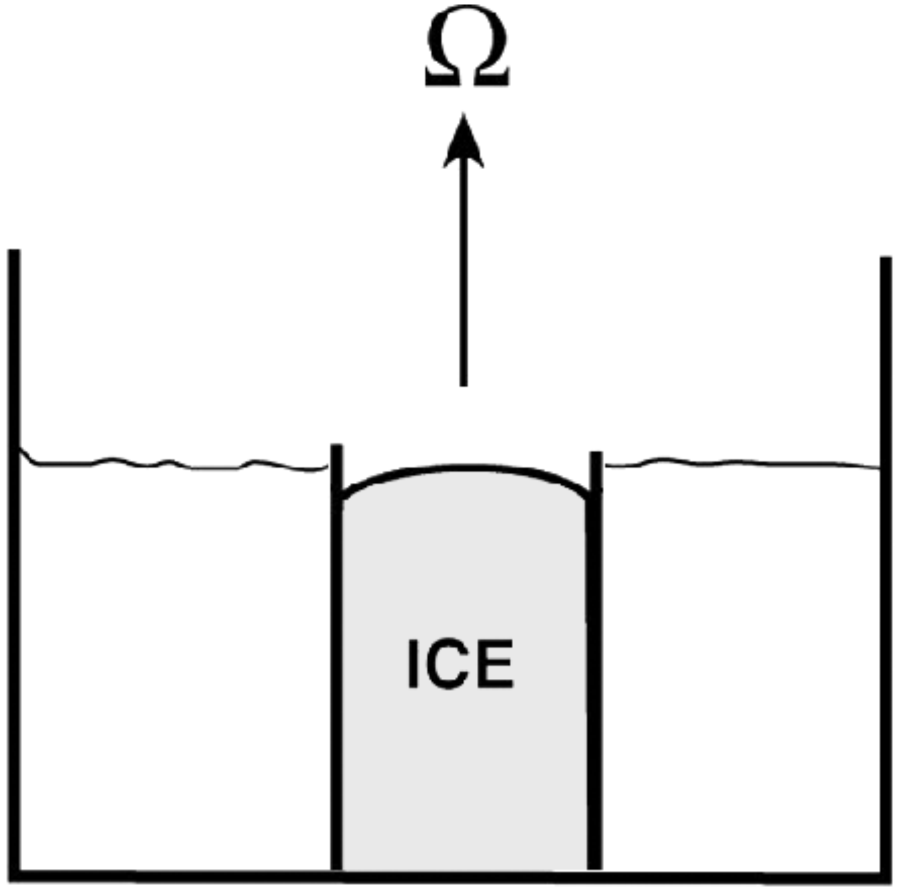
- For the air, we can use the pressure coordinate.

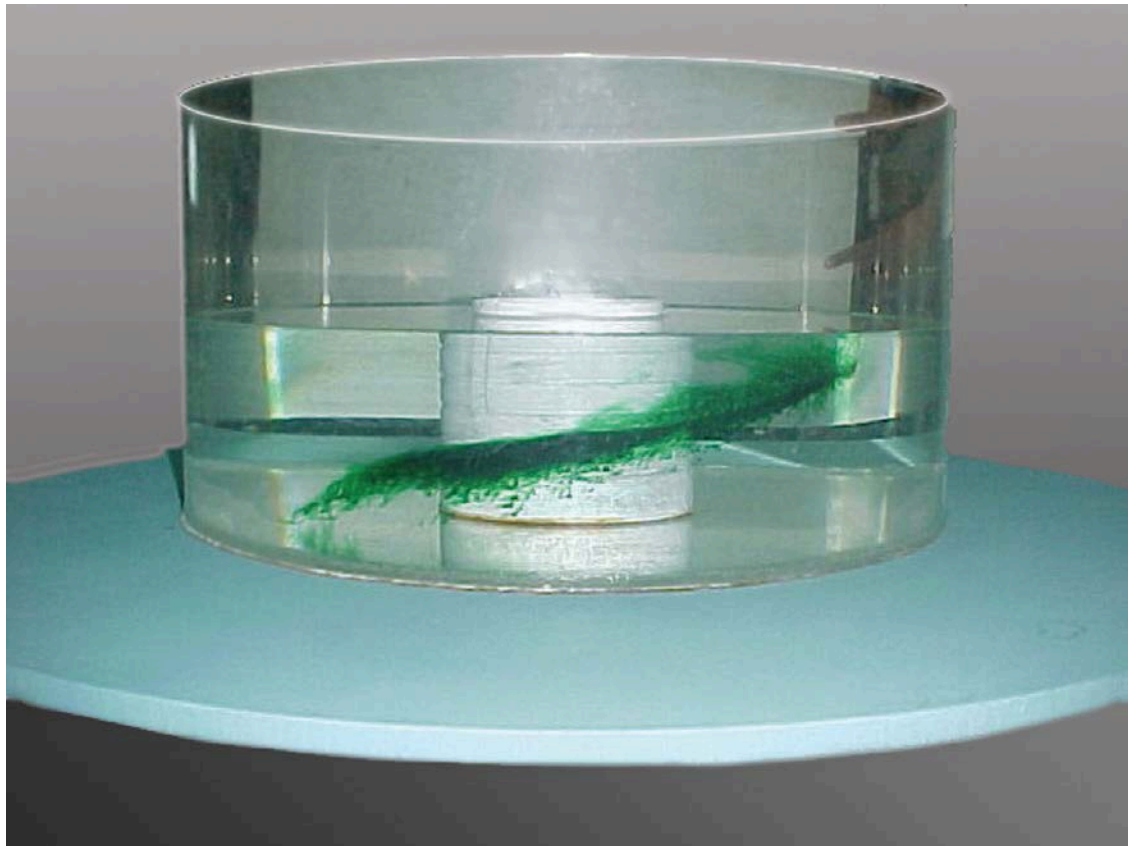
$$\frac{\partial u_g}{\partial p} = -\frac{g}{f}\frac{\partial^2 z}{\partial p \partial y} = -\frac{g}{f}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial p}\right) = \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) = \frac{R}{fp}\frac{\partial T}{\partial y}$$

$$\frac{\partial v_g}{\partial p} = \frac{g}{f}\frac{\partial^2 z}{\partial p \partial x} = \frac{g}{f}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial p}\right) = -\frac{1}{f}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = -\frac{R}{fp}\frac{\partial T}{\partial x}$$

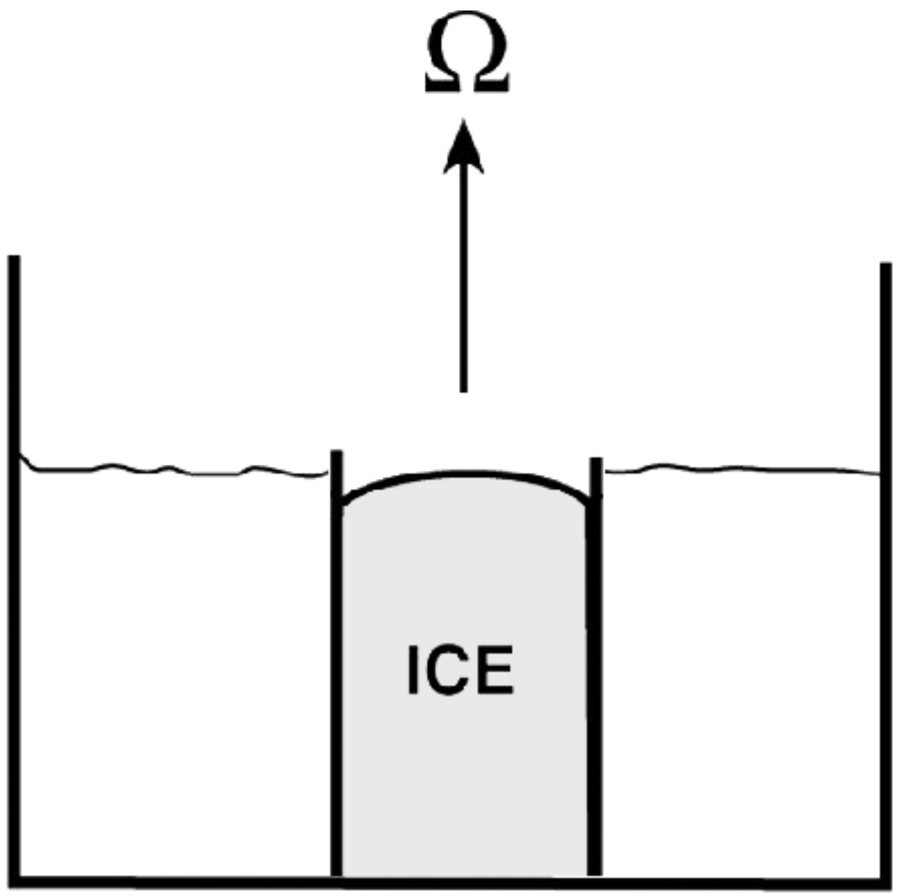
hydrostatic balance   ideal gas law

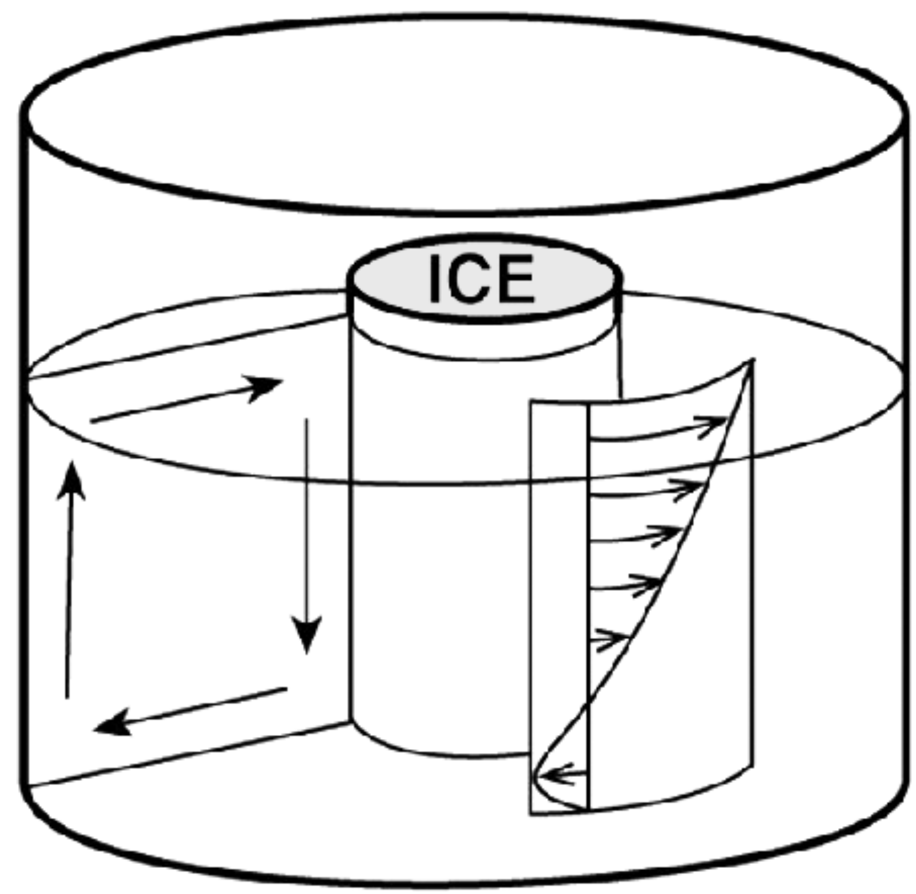
3. Thermal wind equation





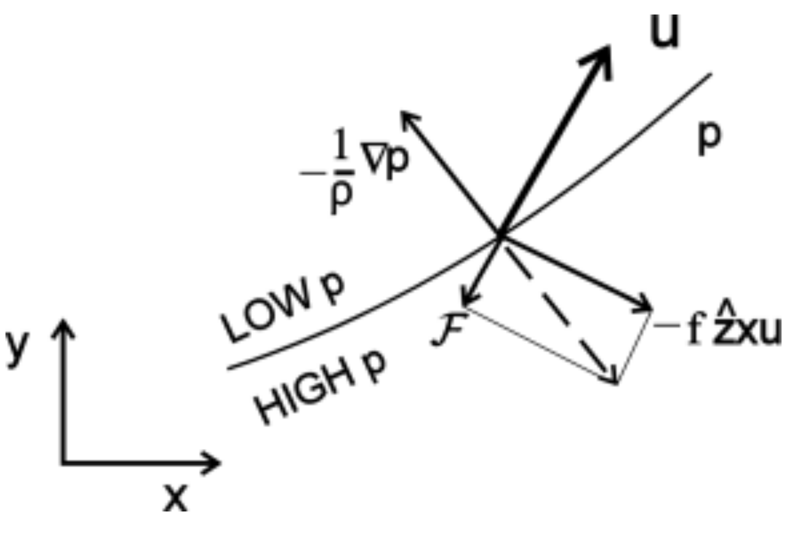
3. Thermal wind equation





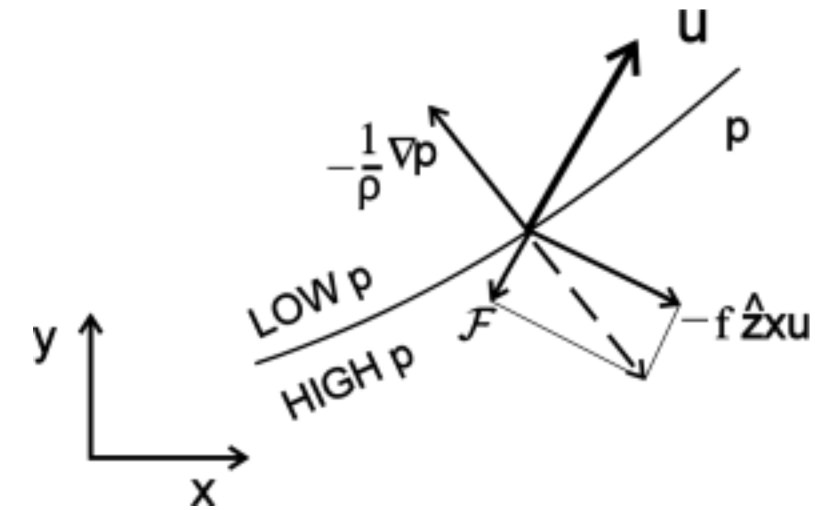
4. Ageostrophic wind

- When F is not negligible,
- $f\hat{\mathbf{z}} \times \mathbf{u} + \frac{1}{\rho} \nabla p = F,$ and F directs to the opposite direction of the flow.
- If we write the horizontal velocity $\mathbf{u}_h = \mathbf{u}_g + \mathbf{u}_{ag}$, then $f\hat{\mathbf{z}} \times \mathbf{u}_{ag} = F \rightarrow \mathbf{u}_{ag}$ is always to the right of F in the northern hemisphere.

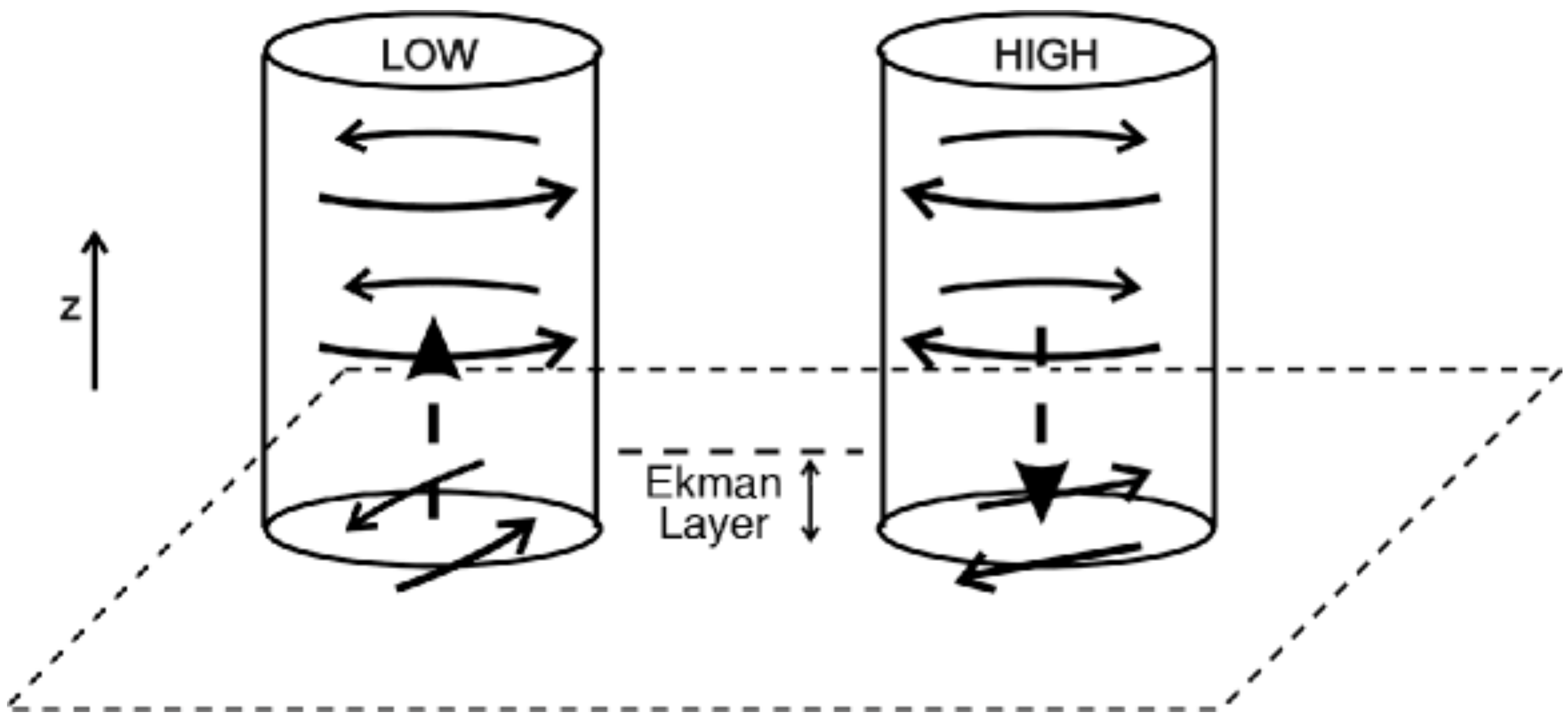


4. Ageostrophic wind

- $F = -\frac{k}{\delta}\mathbf{u}$, where δ is the depth of the Ekman layer and k is a drag coefficient.
- When $\frac{\partial p}{\partial x} = 0$, $-fv = -k\frac{u}{\delta}$ and $fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = -k\frac{v}{\delta}$.
- $u = -\frac{1}{\left(1+\frac{k^2}{f^2\delta^2}\right)}\frac{1}{\rho f}\frac{\partial p}{\partial y} < u_g$
- $\frac{k}{f\rho} \sim 0.1$, so u is slightly smaller then u_g.
- $\frac{v}{u} = \frac{k}{f\delta} \sim 0.1 \rightarrow$ wind and isobar has an angle between 6 to 12 deg.
- Ageostrophic component is larger over land (k is larger over land) and at low latitudes (f is small).



4. Ageostrophic wind



$$\nabla_p \cdot \mathbf{u}_{ag} + \frac{\partial \omega}{\partial p} = 0$$

→ convergence/divergence of ageostrophic wind creates a vertical motion.

Figure 7.26, Marshall and Plumb (2008)

4. Ageostrophic wind

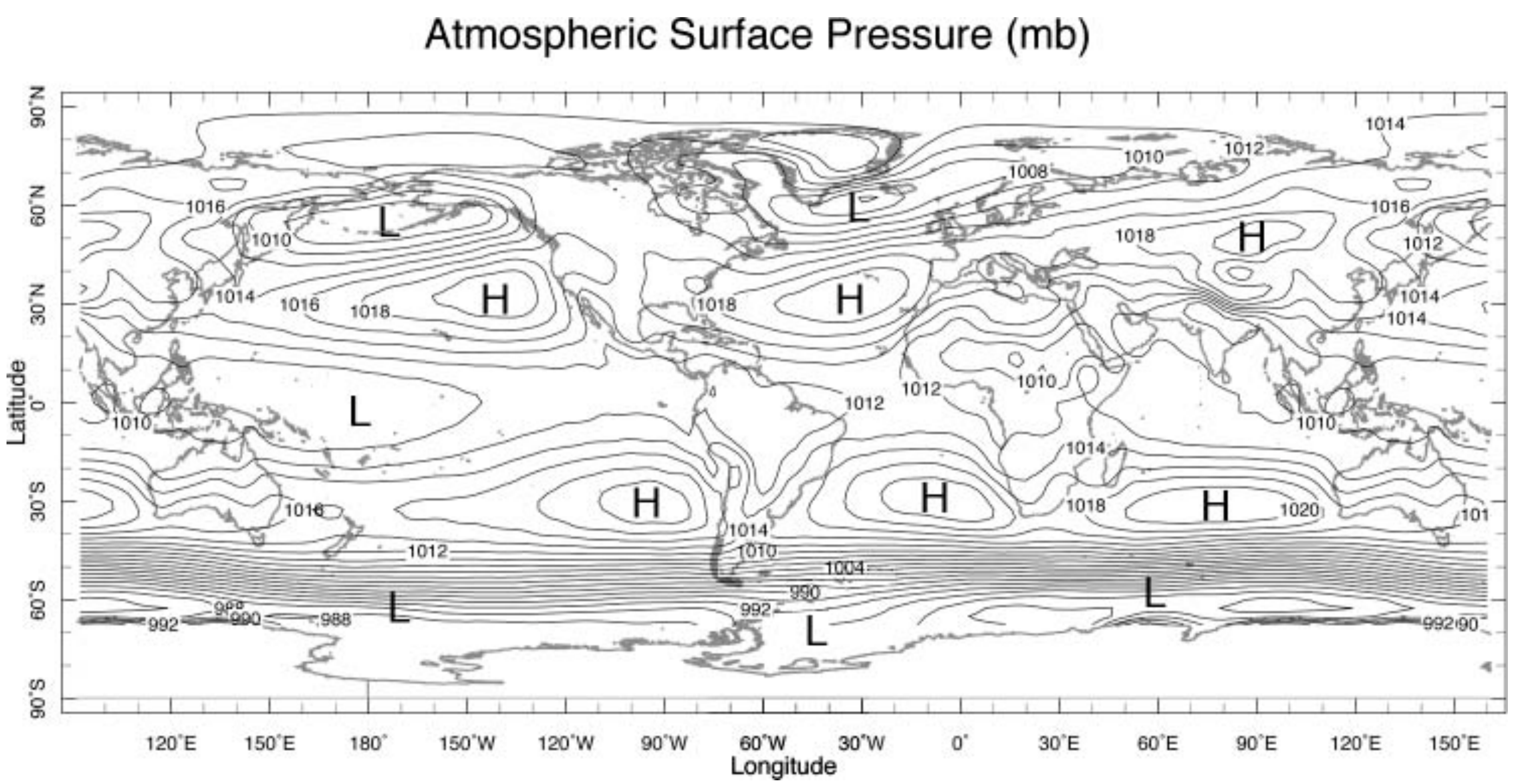


Figure 7.27, Marshall and Plumb (2008)

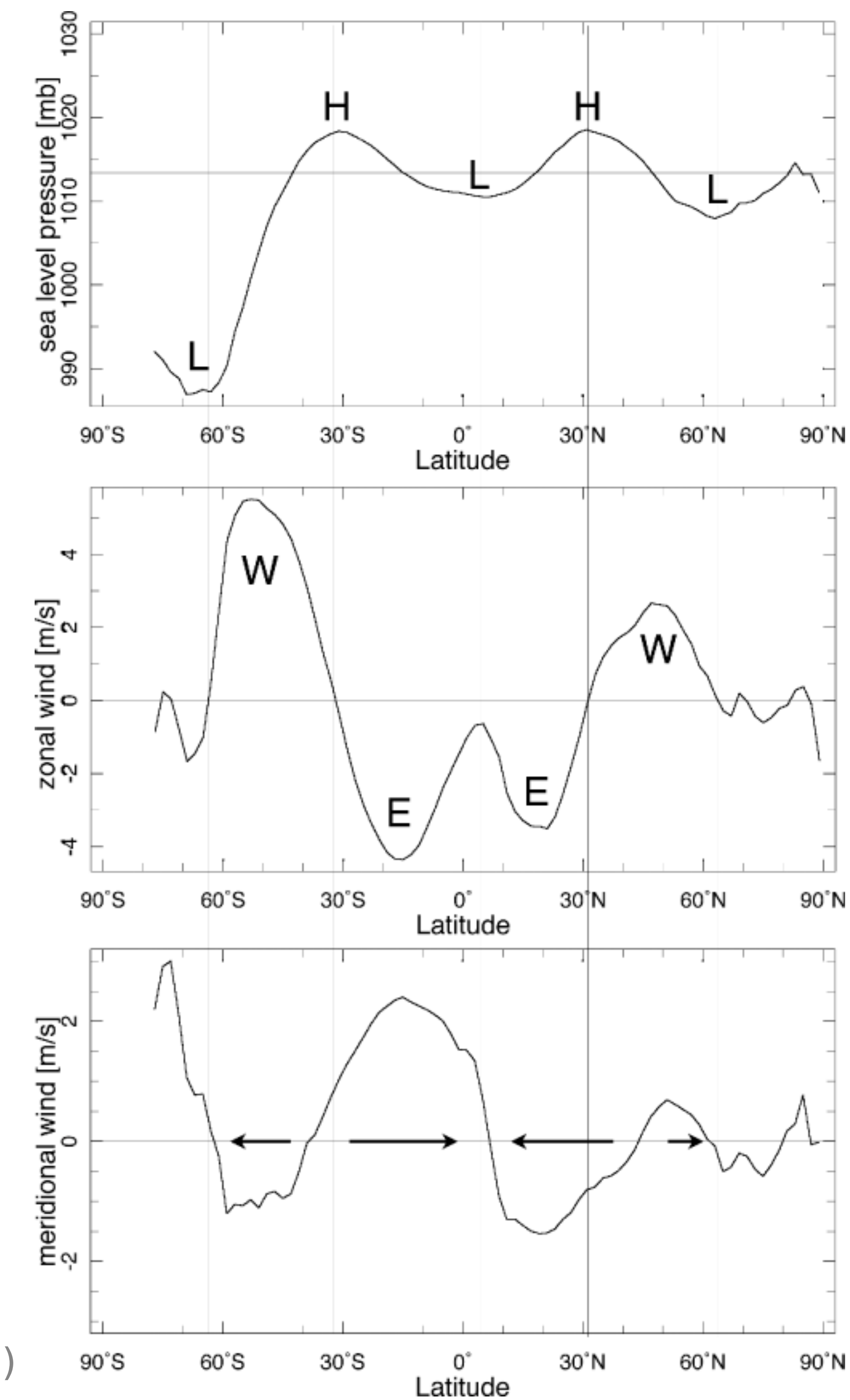


Figure 7.28, Marshall and Plumb (2008)